Глава 2. Некоторые классические задачи математической физики

§5. Уравнение $\Delta u + cu = -f$ в неограниченной области

Парциальные условия излучения ($e^{-i\omega t}$).

$$\int_0^b \left\{ \frac{\partial u}{\partial x} + i\gamma_n^{(1)} u \right\}_{x=0} \psi_n(y)dy = 2i\gamma_{n_0} A_{n_0} \delta_{nn_0} \quad (44) \quad (n = 1, 2, \ldots)$$

Падающая(правая) волна

$$u_{n_0}(x,y) = A_{n_0} e^{i\gamma_{n_0} x} \psi_{n_0}(y) \quad \Rightarrow$$

$$\int_0^b \left\{ \frac{\partial u_{n_0}}{\partial x} + i\gamma_n^{(1)} u_0 \right\}_{x=0} \psi_n(y)dy = \int_0^b \left\{ i\gamma_{n_0}^{(1)} A_{n_0} + i\gamma_n^{(1)} A_{n_0} \right\}_{x=0} \psi_{n_0}(y)\psi_n(y)dy =$$

$$= \left\{ i\gamma_{n_0}^{(1)} A_{n_0} + i\gamma_n^{(1)} A_{n_0} \right\} \boxed{\int_0^b \psi_{n_0}(y)\psi_n(y)dy} = \quad = \delta_{nn_0}$$

$= 2i\gamma_{n_0} A_{n_0} \delta_{nn_0} \quad \Rightarrow$ падающая волна $u_{n_0}(x,y)$ проходит сечение x=0

Любая правая волна при $n \neq n_0$ пройти сечение x=0 не может:

$$\overset{(\rightarrow)}{u_k}(x,y) = A_k e^{i\gamma_k x} \psi_k(y), \quad k \neq n_0 \Rightarrow$$

$$\int_0^b \left\{ \frac{\partial \overset{(\rightarrow)}{u_k}}{\partial x} + i\gamma_n^{(1)} \overset{(\rightarrow)}{u_k} \right\}_{x=0} \psi_n(y)dy = \left\{ i\gamma_k^{(1)} A_k + i\gamma_n^{(1)} A_k \right\}_{x=0} \boxed{\int_0^b \psi_n(y)\psi_k(y)dy} = \quad = \delta_{nk}$$

$$= 2i\gamma_k^{(1)} A_k = 2i\gamma_{n_0}^{(1)} A_{n_0} \underbrace{\boxed{\delta_{kn_0}}}_{=0} = 0 \quad \Rightarrow \quad A_k = 0, k \neq n_0$$

Любая левая волна проходит сечение x=0: $\overset{(\leftarrow)}{u_k}(x,y) = B_k e^{-i\gamma_k x} \psi_k(y) \quad \Rightarrow$

$$\int_0^b \left\{ \frac{\partial \overset{(\leftarrow)}{u_k}}{\partial x} + i\gamma_n^{(1)} \overset{(\leftarrow)}{u_k} \right\}_{x=0} \psi_n(y)dy =$$

$$= \left\{ -i\gamma_k^{(1)} B_k + i\gamma_n^{(1)} B_n \right\}_{x=0} \boxed{\int_0^b \psi_n(y)\psi_k(y)dy} =$$

$$= (-i\gamma_n^{(1)} B_n + i\gamma_n^{(1)} B_n) = 0$$

Квадрупольный излучатель

$$p(r,\theta) = \sum_{n=0}^{\infty} C_n \zeta_n^{(1)}(kr) P_n(\cos\theta) \qquad (61)$$

$$\zeta_n^{(1)}(kr) = \sqrt{\frac{\pi}{2kr}} H_{n+\frac{1}{2}}^{(1)}(kr) \qquad (62)$$

$$\frac{\partial p}{\partial r}\Big|_{r=a} = i\omega\rho_0 v_0 P_2(\cos\theta) \quad (58) \quad \Rightarrow$$

$$\sum_{n=0}^{\infty} C_n k \zeta_n^{(1)\prime}(ka) P_n(\cos\theta) = i\omega\rho_0 v_0 P_2(\cos\theta)$$

$$C_2 k \zeta_2^{(1)\prime}(ka) = i\omega\rho_0 v_0 \quad \Rightarrow$$

$$C_2 = i\overbrace{\frac{\omega}{k}}^{=c} \frac{\rho_0 v_0}{\zeta_2^{(1)\prime}(ka)} = ic\rho_0 v_0 \frac{1}{\zeta_2^{(1)\prime}(ka)} \quad \Rightarrow$$

$$p(r,\theta) = ic\rho_0 v_0 \frac{\zeta_2^{(1)}(kr)}{\zeta_2^{(1)\prime}(ka)} P_2(\cos\theta) \qquad (64)$$

$$v_r = \frac{1}{i\omega\rho_0} \frac{\partial p}{\partial r} \qquad (55) \Rightarrow$$

$$= \omega$$

$$v_r(r,\theta) = \frac{1}{i\omega\rho_0} ic\rho_0 v_0 k \frac{\zeta_2^{(1)\prime}(kr)}{\zeta_2^{(1)\prime}(ka)} P_2(\cos\theta) =$$

$$= v_0 \frac{\zeta_2^{(1)\prime}(kr)}{\zeta_2^{(1)\prime}(ka)} P_2(\cos\theta) \qquad (65)$$

$$ka = \frac{2\pi}{\lambda} a \ll 1, \quad kr = \frac{2\pi}{\lambda} r \gg 1$$

$$p(r,\theta) = ic\rho_0 v_0 \frac{\zeta_2^{(1)}(kr)}{\zeta_2^{(1)\prime}(ka)} P_2(\cos\theta) = ic\rho_0 v_0 \frac{-\frac{1}{kr} e^{ikr} e^{-i\frac{\pi}{2}}}{i\frac{9}{(ka)^4}} P_2(\cos\theta) =$$

$$= \frac{ic\rho_0 v_0 k^4 a^4}{i9kr} e^{ikr} e^{-i\frac{\pi}{2}} P_2(\cos\theta) = -\frac{c\rho_0 v_0 k^3 a^4}{9r} e^{ikr} e^{-i\frac{\pi}{2}} P_2(\cos\theta) \quad (67)$$

$$v_r(r,\theta) = v_0 \frac{\zeta_2^{(1)\prime}(kr)}{\zeta_2^{(1)\prime}(ka)} P_2(\cos\theta) = v_0 \frac{-\frac{i}{kr} e^{ikr} e^{-i\frac{\pi}{2}}}{i\frac{9}{(ka)^4}} P_2(\cos\theta) =$$

$$= -\frac{v_0 k^4 a^4}{9kr} e^{ikr} e^{-i\frac{\pi}{2}} P_2(\cos\theta) = -\frac{v_0 k^3 a^4}{9r} e^{ikr} e^{-i\frac{\pi}{2}} P_2(\cos\theta) \quad (68)$$

$$\overline{P}(r,\theta,t) = \mathrm{Re}(p(r,\theta)e^{-i\omega t}) = \mathrm{Re}\left(-\frac{c\rho_0 v_0 k^3 a^4}{9r} P_2(\cos\theta) e^{-i\left(\omega t - kr + \frac{\pi}{2}\right)} \right) =$$

$$= -\frac{c\rho_0 v_0 k^3 a^4}{9r} P_2(\cos\theta) \cos\left(\omega t - kr + \frac{\pi}{2}\right)$$

$$\overline{V}_r\left(r,\theta,t\right)=\mathrm{Re}(v_r(r,\theta)e^{-i\omega t})=\mathrm{Re}\left(-\frac{v_0k^3a^4}{9r}P_2(\cos\theta)e^{-i\left(\omega t-kr+\frac{\pi}{2}\right)}\right)=$$

$$=-\frac{v_0k^3a^4}{9r}P_2(\cos\theta)\cos\left(\omega t-kr+\frac{\pi}{2}\right)$$

$$\overline{Y}=\frac{1}{T}\int_0^T\overline{P}(r,\theta,t)\overline{V}_r\left(r,\theta,t\right)dt=\frac{c\rho_0v_0^2k^6a^8P_2^2(\cos\theta)}{81r^2}\frac{1}{T}\int_0^T\cos^2\left(\omega t-kr+\frac{\pi}{2}\right)dt=$$

$$\frac{1}{T}\int_0^T\cos^2\left(\omega t-kr+\frac{\pi}{2}\right)dt=\frac{1}{2T}\int_0^T(1+\cos 2\left(\omega t-kr+\frac{\pi}{2}\right))dt=\frac{1}{2}\Rightarrow$$

$$\overline{Y}=\frac{c\rho_0v_0^2k^6a^8P_2^2(\cos\theta)}{162r^2}$$

$$\Pi=r^2\int_0^{2\pi}\int_0^{\pi}\overline{Y}\sin\theta d\theta d\varphi=\frac{2\pi c\rho_0v_0^2k^6a^8}{162}\int_{\pi}^{0}P_2^2(\cos\theta)d\cos\theta=$$

$$=\frac{\pi c\rho_0v_0^2k^6a^8}{81}\int_{-1}^{1}P_2^2(x)dx=\frac{\pi c\rho_0v_0^2k^6a^8}{81}\left\|P_2\right\|^2=$$

$$=\frac{\pi c\rho_0v_0^2k^6a^8}{81}\frac{2}{2\cdot 2+1}=\frac{2\pi c\rho_0v_0^2k^6a^8}{405}$$

$\overline{Y}$ - интенсивность излучения квадруполя,

Π - мощность излучения квадруполя.