Глава VI. Методы исследования математических моделей

Метод малого параметра. Сингулярные возмущения

$$\begin{cases} \mu \dfrac{dy}{dt} = f(y,t), \quad 0 < t \le T, \\ y(0) = y^0, \end{cases} \qquad (17), (18)$$

$$\begin{aligned} \Im = f(y_0(\mu\tau) + \mu\, y_1(\mu\tau) + ... + \Pi_0(\tau) + \mu\Pi_1(\tau) + ..., \mu\tau) - \\ - f(y_0(\mu\tau) + \mu\, y_1(\mu\tau) + ..., \mu\tau), \end{aligned} \qquad (22)$$

$$F = F_0(t) + \mu F_1(t) + \ldots; \quad \Im = \Im_0(\tau) + \mu \Im_1(\tau) + ... \qquad (23)$$

$$F_0(t) = f(y_0(t), t) = 0 \qquad (26)$$

$$\frac{dy_0}{dt} = F_1(t) \qquad (27)$$

$$\frac{d\Pi_0}{d\tau} = \Im_0(\tau) \qquad (28)$$

$$\frac{d\Pi_1}{d\tau} = \Im_1(\tau) \qquad (29)$$

23 22 Так как в (17) μ=0

$$(25) \Rightarrow F_0(t) = F\Big|_{\mu=0} = f(y_0(t);\ t) = 0 \quad (26)$$

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$$F_0(\tau) = F\Big|_{\mu=0} = f(y_0(0) + \Pi_0(\tau), 0) - f(y_0(0), 0) = f(y_0(0) + \Pi_0(\tau), 0) \qquad (30)$$

$$(21) \Rightarrow y(0,\mu) = y_0(0) + \mu y_1(0) + \ldots + \Pi_0(0) + \mu \Pi_1(0) + \ldots =$$

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$$= y^0 = y_0^0 + \mu y_1^0 + \ldots \qquad (31)$$

$$(31) \Rightarrow \Pi_0(0) = y_0^0 - y_0(0) \qquad (32)$$

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$$\begin{aligned}(28) &\Rightarrow \begin{cases} \dfrac{d\Pi_0}{d\tau} = F_0(\tau) = f(y_0(0)+\Pi_0(\tau),0), \tau > 0, & (33) \\ \Pi_0(0) = y_0^0 - y_0(0) & (34)\end{cases} \\ (32) &\Rightarrow \end{aligned}$$

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$$(27) \Rightarrow \frac{dy_0}{dt} = F_1(t) = \frac{\partial F(t)}{\partial \mu}\Big|_{\mu=0} = \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \mu}\Big|_{\mu=0} = f_y(y_0(t),t)y_1(t) \quad (35)$$

$$(29) \Rightarrow \frac{d\Pi_1}{d\tau} = F_1(t) = \frac{\partial F(t)}{\partial \mu}\Big|_{\mu=0} = f_y(y_0(0)+\Pi_0(\tau),0)\frac{\partial y}{\partial \mu}\Big|_{\mu=0} - f_y(y_0(0),0)\frac{\partial y}{\partial \mu}\Big|_{\mu=0} +$$

$$+ f_t(y_0(0)+\Pi_0(\tau),0)\underbrace{\frac{\partial t}{\partial \mu}}_{=\tau}\Big|_{\mu=0} - f_t(y_0(0),0)\frac{\partial t}{\partial \mu}\Big|_{\mu=0} = \quad (=\tau)$$

$$= f_y(y_0(0)+\Pi_0(\tau),0)\left(\left(\frac{\partial y_0(\mu\tau)}{\partial t}\frac{\partial t}{\partial \mu} + y_1(\mu)\right) + \mu\frac{\partial y_1(\mu\tau)}{\partial t}\frac{\partial t}{\partial \mu} + \ldots +\right.$$

$$+ \underline{\Pi_1(\tau)} + 2\mu\Pi_2(\tau))\Big|_{\mu=0} - f_y(y_0(0),0)\left(\frac{\partial y_0(\mu\tau)}{\partial t}\overbrace{\frac{\partial t}{\partial \mu}}^{=\tau} + y_1(\mu) + \mu\frac{\partial y_1(\mu\tau)}{\partial t}\overbrace{\frac{\partial t}{\partial \mu}}^{=\tau}\right)\Big|_{\mu=0} +$$

$$+ f_t(y_0(0)+\Pi_0(\tau),0)\overbrace{\frac{\partial t}{\partial \mu}}^{=\tau}\Big|_{\mu=0} - f_t(y_0(0),0)\overbrace{\frac{\partial t}{\partial \mu}}^{=\tau}\Big|_{\mu=0} = f_y(y_0(0)+\Pi_0(\tau),0)\Pi_1(\tau) +$$

$$\boxed{+(f_y(y_0(0)+\Pi_0(\tau),0) - f_y(y_0(0),0)(y_0'(0)\tau + y_1(0)) + (f_t(y_0(0)+\Pi_0(\tau),0) - f_t(y_0(0),0))\tau} = Q_1$$

$$= f_y(y_0(0)+\Pi_0(\tau),0)\Pi_1(\tau) + Q_1 \qquad (36)$$

$$(31) \Rightarrow y(0,\mu) = y_0(0) + \underline{\underline{\mu y_1(0)}} + \ldots + \Pi_0(0) + \mu\Pi_1(0) + \ldots =$$

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$$= y^0 = y_0^0 + \underline{\underline{\mu y_1^0}} + \ldots \Rightarrow$$

$$\Pi_1(0) = y_1^0 - y_1(0) \qquad (37)$$

Цепочка решения:

Алгебраическое уравнение:

$$F_0(t) = f(y_0(t), t) = 0 \qquad (26)$$

↓

$$y_0(t)$$

↓

Задача Коши:

$$\begin{cases} \dfrac{d\Pi_0}{d\tau} = f(y_0(0) + \Pi_0(\tau), 0), \tau > 0, & (33) \\ \Pi_0(0) = y_0^0 - y_0(0) & (34) \end{cases}$$

↓

$$\Pi_0(\tau)$$

↓

Алгебраическое уравнение:

$$\frac{dy_0}{dt} = f_y(y_0(t), t) y_1(t) \qquad (35)$$

↓

$$y_1(t)$$

↓

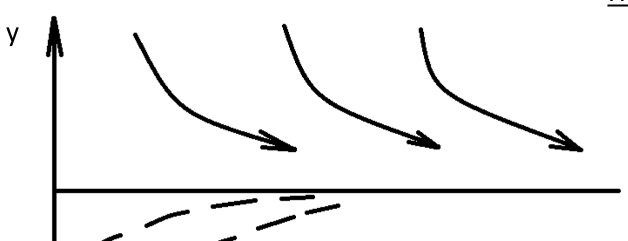
Задача Коши:

$$\begin{cases} \dfrac{d\Pi_1}{d\tau} = f_y(y_0(0) + \Pi_0(\tau), 0)\Pi_1(\tau) + Q_1, \tau > 0, & (36) \\ \Pi_1(0) = y_1^0 - y_1(0) \end{cases}$$

↓

$$\Pi_1(\tau)$$

<u>Пример:</u>



$$\begin{cases} \mu \dfrac{dy}{d\tau} = y - y^2, 0 < t \le 1, & (40) \\ y(0) = \dfrac{1}{2} + \mu = y_0^0 + \mu y_1^0 & (41) \end{cases}$$

1

μ

½

$(26) \Rightarrow f(y_0(t), t) = 0 \Rightarrow$

t

$$y_0(t) - y_0^2(t) = 0 \Rightarrow y_0(t) = 1$$

0

$$\begin{cases} \dfrac{d\Pi_0}{d\tau} = f(\underset{=1}{y_0(0)} + \Pi_0(\tau), 0) \\ \Pi_0(0) = y_0^0 - \underset{=1}{y_0(0)} \end{cases} \Rightarrow \begin{cases} \dfrac{d\Pi_0}{d\tau} = (1+\Pi_0) - (1+\Pi_0)^2, \tau > 0, (42) \\ \Pi_0(0) = \dfrac{1}{2} - 1 = -\dfrac{1}{2} \end{cases}$$

$$\underset{=0}{\boxed{\frac{dy_0}{dt}}} = f_y(y_0(t), t) y_1(t) \quad (35) \quad \Leftarrow \quad \Pi_0(\tau) = -\frac{1}{1+e^{\tau}}$$

$$y_1(t) = 0 \Rightarrow y(t,\mu) = \underset{=1}{y_0(t)} + \mu \underset{=0}{y_1(t)} + \Pi_0(\tau) + \mu \Pi_1(\tau) + \ldots =$$

$$= 1 + \Pi_0(\tau) + \underline{\underline{O}}(\mu) = 1 - \frac{1}{1+e^{\tau}} + \underline{\underline{O}}(\mu) \qquad (44)$$

$$\begin{matrix} (36) \\ (37) \\ (44) \end{matrix} \Rightarrow \begin{cases} \dfrac{d\Pi_1}{d\tau} = (1 - 2(1 - \dfrac{1}{1+e^{\tau}}))\Pi_1, \tau > 0, \\ f_y = 1 - 2y \\ \Pi_1(0) = y_1^0 - y_1(0) = 1 - 0 = 1 \Rightarrow \end{cases}$$

$$\Pi_1(\tau) = \frac{4e^{\tau}}{(1+e^{\tau})^2} \Rightarrow y(t,\mu) = \underset{=1}{y_0(t)} + \mu \underset{=0}{y_1(t)} + \Pi_0(\tau) +$$

$$+ \mu \Pi_1(\tau) + \underline{\underline{O}}(\mu^2) = 1 - \frac{1}{1+e^{\tau}} + \mu \frac{4e^{\tau}}{(1+e^{\tau})^2} + \underline{\underline{O}}(\mu^2) \; (48)$$

Метод усреднения Крылова-Боголюбова

$$\begin{cases} \ddot{y}-\varepsilon(1-y^2)\dot{y}+y=0, \quad t>0, & (6) \\ y(0)=y_0\,,\ \dot{y}(0)=0. & (7) \end{cases}$$

$$\begin{cases} \dot{y}=u\,, \qquad y(0)=y_0, \\ \dot{u}=\varepsilon(1-y^2)u-y, \quad u(0)=0. \end{cases} \qquad (28)$$

$$\begin{aligned} y &= a\cos(t+\theta), \\ u &= -a\sin(t+\theta), \end{aligned} \qquad (29)$$

$$\begin{cases} \dot{y}=\dot{a}\cos(t+\theta)-a\sin(t+\theta)\left(1+\dot{\theta}\right)=-a\sin(t+\theta) \\ \dot{u}=-\dot{a}\sin(t+\theta)-a\cos(t+\theta)\left(1+\dot{\theta}\right)=-\varepsilon\left(1-a^2\cos^2(t+\theta)\right)a\sin(t+\theta)- \\ -a\cos(t+\theta) \end{cases}$$

$$\begin{cases} \cos(t+\theta)\dot{a}-a\sin(t+\theta)\dot{\theta}=0 \\ \sin(t+\theta)\dot{a}+a\cos(t+\theta)\dot{\theta}=\varepsilon\left(1-a^2\cos^2(t+\theta)a\sin(t+\theta)\right) \end{cases}$$

$$\Delta=a\cos^2(t+\theta)+a\sin^2(t+\theta)=a$$

$$\Delta_a = \varepsilon a^2 \left(1 - a^2 \cos^2\left(t+\theta\right)\right) \sin^2\left(t+\theta\right) =$$

$$= \varepsilon a^2 \left(1 - a^2 \cos^2\left(t+\theta\right)\right) \sin^2\left(t+\theta\right) =$$

$$= \varepsilon a^2 \left(\sin^2\left(t+\theta\right) - a^2 \sin^2\left(t+\theta\right) \cos^2\left(t+\theta\right)\right) =$$

$$= \varepsilon a^2 \left(\frac{1-\cos 2\left(t+\theta\right)}{2} - \frac{a^2}{4}\sin^2 2\left(t+\theta\right)\right) =$$

$$= \varepsilon a^2 \left(\frac{1-\cos 2\left(t+\theta\right)}{2} - \frac{a^2}{4}\frac{1-\cos 4\left(t+\theta\right)}{2}\right) =$$

$$= \varepsilon a^2 \left(\frac{1-\cos 2\left(t+\theta\right)}{2} - \frac{a^2}{4}\frac{1-\cos 4\left(t+\theta\right)}{2}\right) =$$

$$= \varepsilon a^2 \left(\frac{1}{2} - \frac{1}{2}\cos 2\left(t+\theta\right) - \frac{a^2}{8} + \frac{a^2}{8}\cos 4\left(t+\theta\right)\right) =$$

$$= \varepsilon a \left(\frac{a}{2}\left(1 - \frac{a^2}{4}\right) - \frac{a}{2}\cos 2\left(t+\theta\right) + \frac{a^3}{8}\cos 4\left(t+\theta\right)\right) \quad \Rightarrow$$

$$\dot{a} = \frac{\Delta_a}{\Delta} = \varepsilon \left(\frac{a}{2}\left(1 - \frac{a^2}{4}\right) - \frac{a}{2}\cos 2\left(t+\theta\right) + \frac{a^3}{8}\cos 4\left(t+\theta\right)\right) \qquad (30a)$$

$$\Delta_\theta = \varepsilon \left(1 - a^2 \cos^2\left(t+\theta\right)\right) a \sin\left(t+\theta\right) \cos\left(t+\theta\right) =$$

$$= \varepsilon \left(1 - a^2 \cos^2\left(t+\theta\right)\right) \frac{a}{2} \sin 2\left(t+\theta\right) =$$

$$= \varepsilon \left(1 - \frac{a^2}{2}\left(1 + \cos 2\left(t+\theta\right)\right)\right) \frac{a}{2} \sin 2\left(t+\theta\right) =$$

$$= \varepsilon\left(\left(\frac{a}{2}-\frac{a^3}{4}\right)\sin 2(t+\theta)-\frac{a^3}{8}\sin 4(t+\theta)\right)=$$

$$= \varepsilon a\left(\frac{1}{2}\left(1-\frac{a^2}{2}\right)\sin 2(t+\theta)-\frac{a^2}{8}\sin 4(t+\theta)\right) \quad \Rightarrow$$

$$\dot{\theta}=\frac{\Delta_\theta}{\Delta}=\varepsilon a\left(\frac{1}{2}\left(1-\frac{a^2}{2}\right)\sin 2(t+\theta)-\frac{a^2}{8}\sin 4(t+\theta)\right) \qquad (30б)$$

$$\begin{cases} \dot{a}=\varepsilon\left\{\frac{a}{2}\left(1-\frac{a^2}{4}\right)-\frac{a}{2}\cos 2(t+\theta)+\frac{a^3}{8}\cos 4(t+\theta)\right\}, \\ \dot{\theta}=\varepsilon\left\{\frac{1}{2}\left(1-\frac{a^2}{2}\right)\sin 2(t+\theta)-\frac{a^2}{8}\sin 4(t+\theta)\right\}, \\ a(0)=y_0, \quad \theta(0)=0. \end{cases} \qquad (30)$$

$$x=\begin{pmatrix} a \\ \theta \end{pmatrix}; \;\; X(x,t)=X_1(x,t); \;\; \xi=\begin{pmatrix} \bar{a} \\ \bar{\theta} \end{pmatrix}; \;\; \dot{x}(t)=\varepsilon X(x,t) \qquad (9)$$

$$X(x,t)=X_1(x,t)+\varepsilon X_2(x,t) \qquad (14)$$

Первое приближение:

$$\dot{\xi}=\varepsilon A_1(\xi); \;\; A_1(\xi)=\bar{X}_1(\xi)$$

$$X_1(x,t)=\begin{pmatrix} \frac{a}{2}\left(1-\frac{a^2}{4}\right)-\frac{a}{2}\cos 2(t+\theta)+\frac{a^3}{8}\cos 4(t+\theta) \\ \frac{1}{2}\left(1-\frac{a^2}{2}\right)\sin 2(t+\theta)-\frac{a^2}{8}\sin 4(t+\theta) \end{pmatrix}$$

$$\bar{X}_1(x,t)=\begin{pmatrix} \frac{\bar{a}}{2}\left(1-\frac{\bar{a}^2}{4}\right) \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{\bar{a}} \\ \dot{\bar{\xi}} \end{pmatrix}=\varepsilon\begin{pmatrix} \frac{\bar{a}}{2}\left(1-\frac{\bar{a}^2}{4}\right) \\ 0 \end{pmatrix} \Rightarrow$$

$$\dot{\bar{a}}=\varepsilon\frac{\bar{a}}{2}\left(1-\frac{\bar{a}^2}{4}\right);\quad \bar{a}(0)=y_0 \quad (31) \qquad \dot{\bar{\theta}}=0;\quad \bar{\theta}(0)=0 \quad (32)$$

$$x_1=\xi \quad\Rightarrow\quad a=\frac{2y_0}{\sqrt{y_0^2+\left(4-y_0^2\right)e^{-\varepsilon t}}};\quad \theta=0$$

$$y=a\cos t=\frac{2y_0\cos t}{\sqrt{y_0^2+\left(4-y_0^2\right)e^{-\varepsilon t}}};\quad y(0)=y_0$$

$$\bar{X}_1(\bar{a})=\frac{\bar{a}}{2}\left(1-\frac{\bar{a}^2}{4}\right)=\frac{\bar{a}}{2}-\frac{\bar{a}^3}{8};\qquad \frac{\partial\bar{X}_1(\bar{a})}{\partial\bar{a}}=\frac{1}{2}-\frac{3\bar{a}^2}{8}$$

Неустойчивый корень:

$$\frac{\partial\bar{X}_1(0)}{\partial\bar{a}}=\frac{1}{2}>0$$

Устойчивый корень:

$$\frac{\partial \bar{X}_1(2)}{\partial \bar{a}} = \frac{1}{2} - \frac{3 \cdot 4}{8} = \frac{1}{2} - \frac{3}{2} = -1 < 0$$

$$\frac{\partial u_1}{\partial t} = X_1(\xi, t) - \bar{X}_1(\xi) \quad (21) \quad \Rightarrow$$

$$\frac{\partial u_1}{\partial t} = \begin{pmatrix} -\frac{a}{2}\cos 2(t+\theta) + \frac{a^3}{8}\cos 4(t+\theta) \\ \frac{1}{2}\left(1 - \frac{a^2}{2}\right)\sin 2(t+\theta) - \frac{a^2}{8}\sin 4(t+\theta) \end{pmatrix} \quad (34)$$