A priori $\to$ A posteriori info update

$$
\begin{tikzcd}[column sep=small]
 & (x_0, F_0) \arrow[d] & & \text{– a priori} \\
(y_1, A_1, S_1) \arrow[r] & \oplus \arrow[r] & (\bar{x}_1, \bar{F}_1) \arrow[dl] & \text{a post after mes. 1.} \\
(y_2, A_2, S_2) \arrow[r] & \oplus \arrow[r] & (\bar{x}_2, \bar{F}_2) & \\
 & \vdots & & \\
 & & (\bar{x}_{k-1}, \bar{F}_{k-1}) \arrow[dl] & \\
(y_k, A_k, S_k) \arrow[r] & \oplus \arrow[r] & (\bar{x}_k, \bar{F}_k) &
\end{tikzcd}
$$

$$\bar{F}_k = \left(\bar{F}_{k-1}^{-1} + A_k^* S_k^{-1} A_k\right)^{-1}$$

$$\bar{x}_k = \bar{F}_k \left(\bar{F}_{k-1}^{-1} \bar{x}_{k-1} + A_k^* S_k^{-1} y_k\right)$$

* Updating is complex
* Deal with info in two different forms.
  - Raw form (y_k, A_k, S_k)
  - Explicit form $(\bar{x}_k, \bar{F}_k)$

Tranform everything to explicit form

$(y_k, A_k, S_k) \longmapsto (x_k, F_k)$ - exp. representation for (y_k, A_k, S_k)

$$F_k = \left(A_k^* S_k^{-1} A_k\right)^{-1}$$

$$x_k = \left(A_k^* S_k^{-1} A_k\right)^{-1} A_k^* S_k^{-1} y_k = F_k A_k^* S_k^{-1} y_k$$

Info. Update in Explicit Form

a priori

$$
\begin{tikzcd}[column sep=small]
 & & & (x_0, F_0) \arrow[r, equal] & (\bar{x}_0, \bar{F}_0) \arrow[d] & \\
\text{Raw info:} & & & & & \\
(y_1, A_1, S_1) \arrow[r, mapsto] & (x_1, F_1) \arrow[rrr] & & & \oplus \arrow[r, mapsto] & (\bar{x}_1, \bar{F}_1) \arrow[dl] \\
(y_2, A_2, S_2) \arrow[r, mapsto] & (x_2, F_2) \arrow[rrr] & & & \oplus \arrow[r, mapsto] & (\bar{x}_2, \bar{F}_2) \\
 & & & & \vdots &
\end{tikzcd}
$$

$$\bar{F}_k = \Big(\bar{F}_{k-1}^{-1} + \underbrace{A_k^* S_k^{-1} A_k}_{= F_k^{-1}}\Big)^{-1} = \underline{\left(\bar{F}_{k-1}^{-1} + F_k^{-1}\right)^{-1}}$$

$$\bar{x}_k = \bar{F}_k\left(\bar{F}_{k-1}^{-1}\, \bar{x}_{k-1} + A_k^* S_k^{-1} y_k\right) =$$

$$= \bar{F}_k\Big(\bar{F}_{k-1}^{-1}\, \bar{x}_{k-1} + F_k^{-1} \underbrace{F_k A_k^* S_k^{-1} y_k}_{= x_k}\Big)$$

$$= \underline{\bar{F}_k\left(\bar{F}_{k-1}^{-1}\, \bar{x}_{k-1} + F_k^{-1} x_k\right)}$$

+ Update ! composition of info representation of the same nature.
+ The most intuitive representation of info.
- Extensive computations at each step.
- Conversion raw $\mapsto$ explicit is not always possible.

It would be more natural to work with F_k^{-1} and $\bar{F}_k^{-1}$

can info for (y_k, A_k, S_k)

(T_k, z_k) $\quad T_k = A_k^* S_k^{-1} A_k = F_k^{-1}$

$$z_k = S_k^{-1} A_k y_k$$

Info update in Canonical Form

a Priori info

$$(x_0, F_0) \longmapsto (T_0, z_0) = (\bar{T}_0, \bar{z}_0)$$

Raw:

$$(y_1, A_1, S_1) \longmapsto (T_1, z_1) \oplus \to (\bar{T}_1, \bar{z}_1)$$

$$(y_2, A_2, S_2) \longmapsto (T_2, z_2) \oplus \to (\bar{T}_2, \bar{z}_2)$$

$$\vdots$$

* Conversion:

* Explicit $\mapsto$ can: $T_0 = F_0^{-1}$

$$z_0 = T_0 x_0$$

* Raw $\mapsto$ can. $T_k = A_k^* S_k^{-1} A_k$

$$z_k = A_k^* S_k^{-1} y_k$$

* Composition $\bar{T}_k = \bar{T}_{k-1} + T_k$

$$\bar{z}_k = \bar{z}_{k-1} + z_k$$

* Estimate $\hat{x}_k = \bar{T}_k^{-1} \bar{z}_k$

$$\mathrm{Var}(\hat{x}_k) = \bar{T}_k^{-1}$$

Working with info in various forms.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
 & \text{"Raw"}\ (y,\ A,\ S) \arrow[dl, shift right, "{\substack{x_0=(A^*S^{-1}A)^{-1}A^*S^{-1}y \\ F=A^*S^{-1}A}}"'] \arrow[dr, "{\substack{T=A^*S^{-1}A \\ z=A^*S^{-1}y}}"] & \\
(x_0, F)\ \text{Expl.} \arrow[ur, dashed, shift right, "{\substack{y=x_0 \\ A=I \\ S=F}}"'] \arrow[rr, shift left, "{T=F^{-1}\quad z=F^{-1}x_0}"] & & (T, z)\ \text{canon.} \arrow[ll, dashed, shift left, "{x_0=T^{-1}z,\ F=T^{-1}}"]
\end{tikzcd}
$$

Composition Operations.

(a) Raw.

$$(y_1, A_1, S_1)\oplus(y_2, A_2, S_2) = \left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}\right)$$

(b) Explicit

$$(x_1, F_1)\oplus(x_2, F_2) = \left((F_1^{-1}+F_2^{-1})^{-1}(F_1^{-1}x_1+F_2^{-1}x_2),\ (F_1^{-1}+F_2^{-1})^{-1}\right)$$

(c) canon.

$$(T_1, z_1)\oplus(T_2, z_2) = (T_1+T_2,\ z_1+z_2)$$

(a) Raw

+ Can always repr in such form
− Size grows
∓ Rel. easy to combine, but size problems
− Producing an est $\hat{x}$ − challenging or impossible.

(b) Explicit

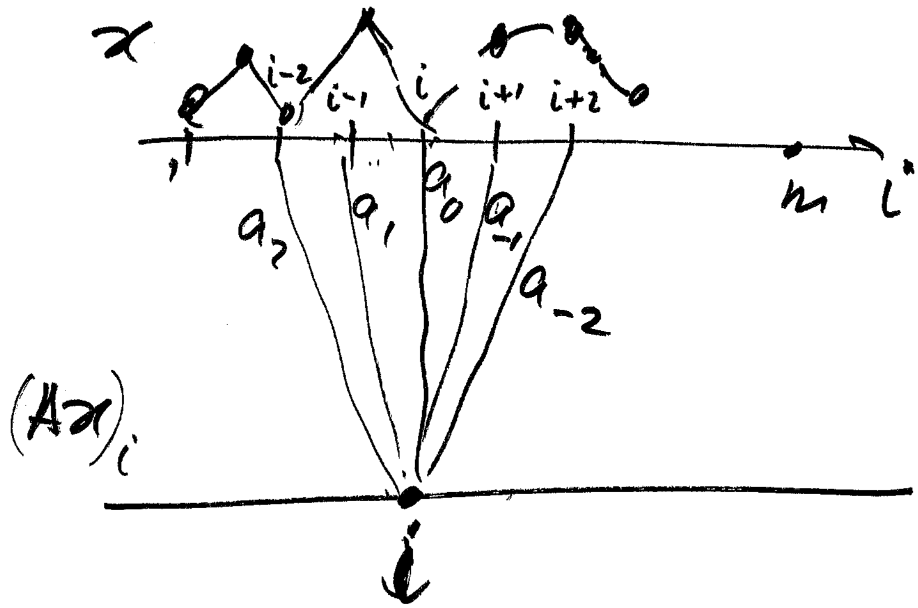
− can be computed only when $A^* S^{-1} A$ is inventible
+ Storage size is const.
− Combining is hard.
+ Getting $\hat{x}$ is trivial.

(c) Canonical

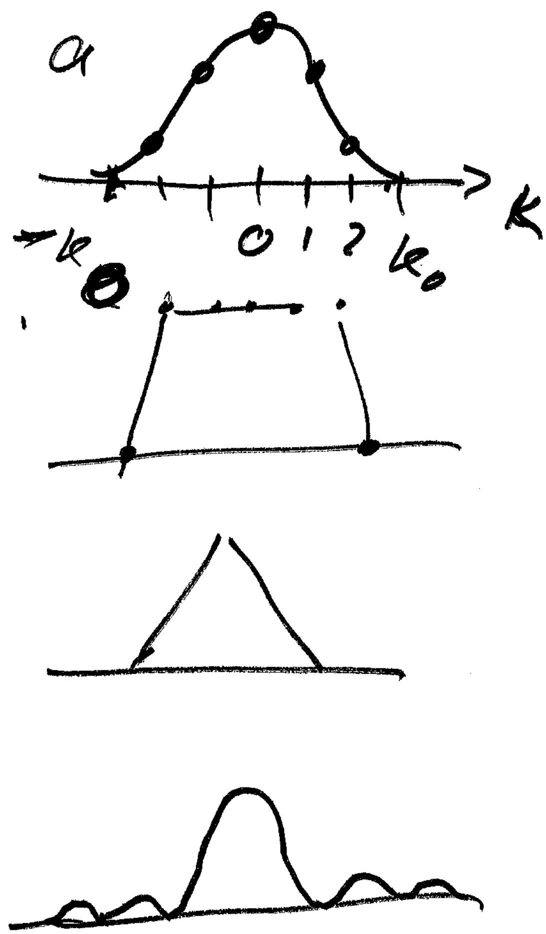
+ Can be always computed
+ ~~Stora~~ Size − const (as in (b))
+ Extremely easy to combine.
± Producing an est $\hat{x}$ is relatively simple.

HW # 4 Comments

Matrix A.

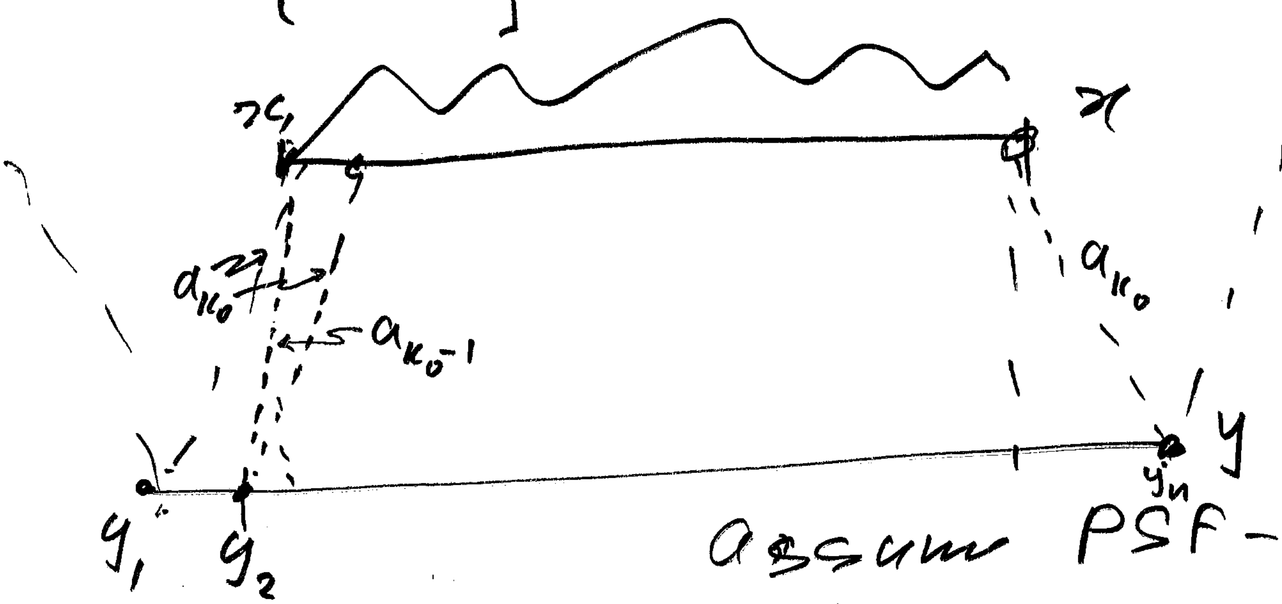


$\mathcal{D} = \mathbb{R}^m$



$$(Ax)_i = \sum_k a_k x_{i-k}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$



assum PSF-symm

$$
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \\ \\ y_n \end{bmatrix}
=
\begin{array}{c}
k_0 \\ \\ \\ \\ k_0 \\ \\
\end{array}
\underbrace{\begin{bmatrix}
a_{k_0} & 0 & \cdots & \cdots & 0 \\
a_{k_0-1} & a_{k_0} & \ddots & & \vdots \\
a_{k_0-2} & \ddots & \ddots & \ddots & \\
\vdots & & \ddots & \ddots & a_{k_0} \\
a_0 & \ddots & & \ddots & a_{k_0-1} \\
\vdots & \ddots & \ddots & & \vdots \\
a_{k_0} & \ddots & & \ddots & a_0 \\
0 & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & a_{k_0}
\end{bmatrix}}_{m}
\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}
+
\begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix}
$$

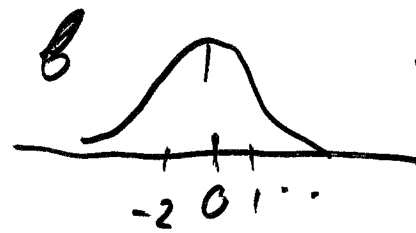
$$n = m - 1 + 2k_0 + 1 = m + 2k_0$$

more **Comments for HW #4.**

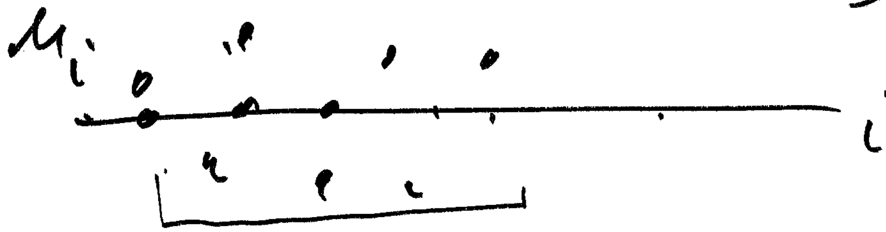
* Generate random signal x
  : $x \sim (0, F)$
* generate $\mu \sim (0, I)$

$\mu_i \sim (0, 1)$

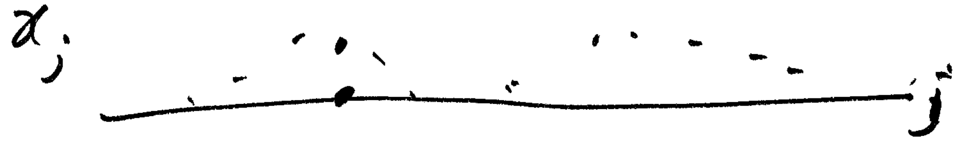
$B\mu$



Point Spread function of sliding window.



$$B = \begin{pmatrix} b_0 & b_1 & \cdots & b_k & 0 & 0 \\ b_1 & b_0 & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ b_k & & \ddots & \ddots & \ddots & \\ 0 & & & \ddots & \ddots & b_1 \\ 0 & & & & b_1 & b_0 \end{pmatrix}$$



Toeplitz matrix.

$x = B\mu \qquad Ex = 0$

$F = \mathrm{Var}\, x = \mathrm{Var}\, B\mu = B \cdot I \cdot B^T = BB^T$

General Least Squares Approach.

$$y = Ax + v \qquad v \sim (0, S)$$

$\|y - A\hat{x}\|^2 \sim \min_{\hat{x}} \to \hat{x}$

work Ok if $S = \sigma^2 I$

$$By = BAx + Bv$$

$$\bar{y} = \bar{A}x + \bar{v}$$

$$\bar{y} = By, \quad \bar{A} = BA, \quad \bar{v} = Bv$$

$$(y, A, S) \mapsto (\bar{y}, \bar{A}, \bar{S})$$

$$\bar{S} = \mathrm{Var}(Bv) = BSB^*$$

Take $B = S^{-1/2}$

$$\bar{S} = S^{-1/2} S S^{-1/2} = I$$

$$\bar{y} = \bar{A}x + \bar{v} \qquad \bar{v} \sim (0, I)$$

$\Rightarrow$ can use LS. approach.

$$Q(x) = \|\bar{y} - \bar{A}x\|^2 =$$

$$= \|S^{-1/2}(y - Ax)\|^2$$

$$= \langle S^{-1/2}(y - Ax), S^{-1/2}(y - Ax) \rangle$$

$$= \langle S^{-1}(y - Ax), y - Ax \rangle$$

$$= \langle S^{-1}y, y \rangle - 2\langle S^{-1}y, Ax \rangle + \langle S^{-1}Ax, Ax \rangle$$

$$= \langle \underbrace{A^*S^{-1}A}_{=T} x, x \rangle - 2\langle \underbrace{A^*S^{-1}y}_{=z}, x \rangle + \langle S^{-1}y, y \rangle$$

$$= \underbrace{\langle Tx, x \rangle - 2\langle z, x \rangle}_{} + \langle S^{-1}y, y \rangle$$

$$\|T^{1/2}(x - T^{-1}z)\|^2 =$$

$$= \langle T^{1/2}x, T^{1/2}x \rangle - 2\langle T^{1/2}x, T^{1/2}T^{-1}z \rangle + \langle T^{1/2}T^{-1}z, T^{1/2}T^{-1}z \rangle$$

$$= \underbrace{\langle Tx, x \rangle - 2\langle z, x \rangle}_{} + \langle T^{-1}z, z \rangle$$

$$Q(x) = \|T^{1/2}(x - T^{-1}z)\|^2 +$$

$$+ \langle S^{-1}y, y\rangle - \langle T^{-1}z, z\rangle$$

$Q = \min$ iff $\hat{x} = T^{-1}z$

$$Q_{min} = \langle S^{-1}y, y\rangle - \langle T^{-1}z, z\rangle$$

$$\hat{x} = T^{-1}z = (A^* S^{-1} A)^{-1} A^* S^{-1} y$$

The same as in the Best Lin Unb. Est.

Gauss - Markov Theorem.

<u>LSE:</u> $\hat{x} = \arg\min \|S^{-1/2}(y - Ax)\|^2$

<u>BLUE</u> $R = \arg\min\{E\|Ry - x\|^2 / ERy = x\}$

$\hat{x} = Ry$

Linear Estimation with the unknown scale of noise

$$y = Ax + v \qquad Var(v) = \underset{\sim}{\sigma^2} S$$

σ^2 is unknown.

$$(y, A, \underset{\sim}{\sigma^2} S)$$

$$\hat{x} = (A^*(\sigma^2 S)^{-1} A)^{-1} A^* (\sigma^2 S)^{-1} y$$

$$= \sigma^2 \cdot \sigma^{-2} (\ )^{-1} \cdots$$

$$= (A^* S^{-1} A)^{-1} A^* S^{-1} y$$

$$Var(\hat{x}) = (A^*(\sigma^2 S)^{-1} A)^{-1} = \underset{\sim}{\sigma^2} (A^* S^{-1} A)^{-1}$$

Need to estimate σ^2 : $\hat{\sigma}^2$

$$Q(x) = \| \bar{y} - \bar{A} x \|^2$$

$$\bar{y} = \bar{A} x + v \qquad \bar{y} = S^{-1/2} y \qquad \bar{A} = S^{-1/2} A$$

$$\bar{v} = S^{-1/2} v \qquad \bar{v} \sim (0, \underset{\sim}{\sigma^2} I)$$