Example (a) #11

$y = Ax + v \qquad v \sim (0, \sigma^2 I)$

$x \in \mathbb{R}^2 \quad x \sim (x_0, F)$

$y_1 = (1+\varepsilon_1)x_1 + v_1$

$y_2 = (1+\varepsilon_2)x_2 + v_2$

$x_0 = \begin{bmatrix} c \\ c \end{bmatrix}, \quad F = \varphi^2 I$

ε_i i.i.d $\quad \varepsilon_i \sim (0, \delta^2)$

$$A = \begin{bmatrix} 1+\varepsilon_1 & 0 \\ 0 & 1+\varepsilon_2 \end{bmatrix} = I + \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix} = A_0 + A'$$

$J = E\, A' \bar{F} A'^*$

$\bar{F} = F + x_0 x_0^T$ – the second moment of x

$\bar{F} = E\, x x^T$

$J = (\varphi^2 + c^2)\delta^2 I$

$$Q = \left(A_0^*(S+J)^{-1}A_0 + F^{-1}\right)^{-1} \quad \Big|\, \mathrm{Var}(\hat{x})$$

$$= \left(I\left(\left(\sigma^2 + (\varphi^2+c^2)\delta^2\right)I\right)^{-1} I + (\varphi^2 I)^{-1}\right)^{-1}$$

$$= \boxed{\frac{1}{\frac{1}{\sigma^2 + (\varphi^2+c^2)\delta^2} + \frac{1}{\varphi^2}}}\, I \longrightarrow \mathrm{Var}\,\hat{x}_1 = \mathrm{Var}\,\hat{x}_2$$

(b) $y_1 = (1+\varepsilon)x_1 + \nu_1 \qquad \varepsilon \sim (0, \delta^2)$

$y_2 = (1+\varepsilon)x_2 + \nu_2$

$$A = (1+\varepsilon)I = \underbrace{I}_{=A_0} + \underbrace{\varepsilon I}_{A'} =$$

$$J = EA'\bar{F}A'^* = E\,\varepsilon I\left[\varphi^2 I + c^2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right]_s \varepsilon I$$

$$= \underbrace{E\varepsilon^2}_{=\delta^2}\cdot\begin{bmatrix}\varphi^2+c^2 & c^2\\ c^2 & \varphi^2+c^2\end{bmatrix}$$

$$J = \delta^2\begin{bmatrix}\varphi^2+c^2 & c^2\\ c^2 & \varphi^2+c^2\end{bmatrix}$$

in (a) $J = \delta^2(\varphi^2+c^2)I = \delta^2\begin{bmatrix}\varphi^2+c^2 & 0\\ 0 & \varphi^2+c^2\end{bmatrix}$

Accumulating Information from Experiments with random A.

$y = Ax + \nu$

Apriori info about x

$x \sim (x_0, F)$

$\nu \sim (0, S)$

$A_0 = EA$

$J = E(A - A_0)F(A - A_0)^*$

Observation (y, A_0, J, S)

$$Q = (F^{-1} + A_0^*(S+J)^{-1}A_0)^{-1}$$

$$\hat{x} = Q(F^{-1}x_0 + A_0^*(S+J)^{-1}y)$$

$y_1 = A_1 x + \nu_1 \qquad (y_1, \bar{A}_1, J_1, S_1)$

$y_2 = A_2 x + \nu_2 \qquad (y_2, \bar{A}_2, J_2, S_2)$

$\bar{A}_1 = EA_1 \qquad J_1 = EA_1' F A_1'^*$

$A_1' = A_1 - \bar{A}_1$

Same for $\bar{A}_2$ J_2

Measurements are indep

$A_1, A_2, \nu_1, \nu_2, x$ are independent.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x + \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}$$

$$y = Ax + \nu \qquad Var(\nu) = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$$

$$\bar{A} = EA = E\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} EA_1 \\ EA_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \end{bmatrix}$$

$$J = ?$$

$$J = E A' \bar{F} A'^*$$

$$A' = A - \bar{A} = \begin{bmatrix} A_1 - \bar{A}_1 \\ A_2 - \bar{A}_2 \end{bmatrix} = \begin{bmatrix} A_1' \\ A_2' \end{bmatrix}$$

$$= E\begin{bmatrix} A_1' \\ A_2' \end{bmatrix} \bar{F} \begin{bmatrix} A_1'^* & A_2'^* \end{bmatrix}$$

$$= E\begin{bmatrix} A_1' \bar{F} A_1'^* & A_1' \bar{F} A_2'^* \\ A_2' \bar{F} A_1'^* & A_2' \bar{F} A_2'^* \end{bmatrix}$$

$$E A_1' \bar{F} A_1'^* = J_1$$

$$E A_1' \bar{F} A_2'^* = \underbrace{EA_1'}_{=0} \cdot \bar{F} \cdot \underbrace{EA_2'^*}_{=0}$$

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

$$S + J = \begin{bmatrix} S_1 + J_1 & 0 \\ 0 & S_2 + J_2 \end{bmatrix}$$

Formulas are the same as for known A. Need to replace

$$S_i \to S_i + J_i \quad A_i \to \bar{A}_i$$

<u>Collecting canonical info</u> (T, z)

<u>apriori inf</u>: $x \sim (x_0, F) \longmapsto T_0 = F^{-1}, z_0 = F^{-1}x_0$

$$(y_1, \bar{A}_1, J_1, S_1) \longmapsto T_1 = \bar{A}_1^*(S_1 + J_1)^{-1}\bar{A}_1$$

$$z_1 = \bar{A}_1^*(S_1 + J_1)^{-1} y_1$$

$$\vdots$$

$$(y_i, \bar{A}_i, J_i, S_i) \longmapsto (T_i, z_i)$$

$$(T, z) = \bigoplus_{i=0}^{n} (T_i, z_i) = \left(\sum_{i=0}^{n} T_i, \sum_{i=0}^{n} z_i\right).$$

Examples

$$y_i = (a + \varepsilon_i)x + \vartheta_i \qquad x \sim (x_0, \varphi^2)$$

a priori inf.

$\varepsilon_i, \vartheta_i \quad i = 1 \cdots n$

independent

$\vartheta_i \sim (0, \sigma^2)$ i.i.d.

$\varepsilon_i \sim (0, \delta^2)$ i.i.d.

$$J_i = E A_i' \bar{F} A_i'^*$$
$$= E\, \varepsilon_i (\varphi^2 + x_0^2) \varepsilon_i$$
$$= (\varphi^2 + x_0^2)\delta^2$$

$$\bar{F} = F + x_0 x_0^T = $$
$$= \varphi^2 + x_0^2$$

$$\begin{cases} T_i = \bar{A}_i^* (S_i + J_i)^{-1} \bar{A}_i = a^2(\sigma^2 + (\varphi^2 + x_0^2)\delta^2)^{-1} \\ z_i = \bar{A}_i^* (S_i + J_i)^{-1} y_i = a(\sigma^2 + (\varphi^2 + x_0^2)\delta^2)^{-1} y_i \end{cases}$$

$$y_i \longmapsto (T_i, z_i)$$

$$(x_0, \varphi^2) \longmapsto (T_0, z_0) = (F^{-1}, F^{-1}x_0) = (\varphi^{-2}, \varphi^{-2}x_0)$$

$$T = T_0 + T_1 + \cdots + T_n$$
$$= \frac{1}{\varphi_0^2} + na^2\Big[\sigma^2 + \underbrace{(\varphi^2 + x_0^2)\delta^2}_{\lambda^2}\Big]^{-1}$$
$$= \frac{1}{\varphi_0^2} + \frac{na^2}{\sigma^2 + \lambda^2} \to \infty$$

$n \to \infty$

$$z = z_0 + z_1 + \cdots z_n$$

$$\mathrm{Var}\, \hat{x} = T^{-1} \to 0$$

Assume that A does not change. Remains same, but unknown.

$$y_i = (a + \varepsilon)x + v_i$$

measurements are <u>not</u> indep.

Combining experiments with same A.

$$y_1 = Ax + v_1 \qquad E A = \underline{A_0}$$
$$y_2 = Ax + v_2 \qquad \underline{J_0} = E A' \bar{F} A'^*$$
$$v_0 \sim (0, S) \text{ i.i.d.}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A \\ A \end{bmatrix} x + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\operatorname{Var}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$$

$$J = E \begin{bmatrix} A' \\ A' \end{bmatrix} \bar{F} \begin{bmatrix} A'^* & A'^* \end{bmatrix}$$

$$= \begin{bmatrix} E A' \bar{F} A'^* & E A' \bar{F} A'^* \\ E A' F A'^* & E A' F A'^* \end{bmatrix} = \begin{bmatrix} J_0 & J_0 \\ J_0 & J_0 \end{bmatrix}$$

Back to example.

$$J_0 = (\varphi^2 + x_0^2)\delta^2 = \lambda^2 \quad 1\times 1 \text{ matrix.}$$

$$J_{n\times n} = \begin{bmatrix} J_0 & J_0 & \cdots & J_0 \\ \cdot & - & - & \cdot \\ J_0 & J_0 & & J_0 \end{bmatrix} = \lambda^2 \begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{bmatrix}$$

$$= \lambda^2 \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{=e} \underbrace{\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}}_{e^T} = \lambda^2 e e^T$$

$$(S+J)^{-1} = (\sigma^2 I + \lambda^2 e e^T)^{-1}$$

$$(\alpha I + \beta e e^T)^{-1} = ? \qquad \begin{bmatrix} \alpha+\beta & & \beta \\ & \ddots & \\ \beta & & \alpha+\beta \end{bmatrix}$$

$$? = \underline{\gamma} I + \underline{\delta} e e^T$$

$$\underset{\text{want}}{I} = (\alpha I + \beta e e^T)(\gamma I + \delta e e^T)$$

$$= \alpha\gamma I + \alpha\delta e e^T + \beta\gamma e e^T + \beta\delta e \underbrace{e^T e}_{=n} e^T$$

$$e^T e = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = n$$

$$= \alpha\underline{\gamma} I + (\alpha \underline{\delta} + \beta\gamma + \beta \underline{\delta} n) e e^T$$

$$\Rightarrow \quad \alpha\gamma = 1 \qquad \Rightarrow \gamma = 1/\alpha$$

$$\alpha\delta + \beta\gamma + \beta\delta n = 0$$

$$(\alpha+\beta n)\delta = -\beta\gamma$$

$$\delta = -\frac{\beta\gamma}{\alpha+\beta n} = -\frac{\beta}{\alpha(\alpha+\beta n)}$$

$$(\alpha I + \beta e e^T)^{-1} = \frac{1}{\alpha}\left(I - \frac{\beta}{\alpha+\beta n} e e^T\right)$$

$$(S+J)^{-1} = (\sigma^2 I + \lambda^2 e e^T)^{-1}$$

$$= \frac{1}{\sigma^2}\left(I - \frac{\lambda^2}{\sigma^2+n\lambda^2} e e^T\right)$$

$$T = A_0^*(S+J)^{-1}A_0 + F^{-1}$$

$$\Big|\ F^{-1} = 1/\varphi^2 \quad A_0 = \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix} = a e$$

$$T = a\, \underline{e}^T \frac{1}{\sigma^2}\left(I - \frac{\lambda^2}{\sigma^2+n\lambda^2} e e^T\right) a \underline{e} + 1/\varphi^2$$

$$= \frac{a^2}{\sigma^2}\left(\underbrace{e^T e}_{=n} - \frac{\lambda^2}{\sigma^2+n\lambda^2} \underbrace{e^T e}_{=n}\underbrace{e^T e}_{=n}\right) + \frac{1}{\varphi^2}$$

$$= n\frac{a^2}{\sigma^2}\left(1 - \frac{n\lambda^2}{\sigma^2+n\lambda^2}\right) + \frac{1}{\varphi^2}$$

$$= n\frac{a^2}{\sigma^2}\,\frac{\sigma^2+n\lambda^2-n\lambda^2}{\sigma^2+n\lambda^2} + \frac{1}{\varphi^2} = \frac{n a^2}{\sigma^2+n\lambda^2} + \frac{1}{\varphi^2}$$

$$T = \frac{n a^2}{\sigma^2 + n\lambda^2} + \frac{1}{\varphi^2} \longrightarrow \frac{a^2}{\lambda^2} + \frac{1}{\varphi^2}$$

as $n \to \infty$

$$Q = \operatorname{var}(\hat{x}) \to \frac{1}{a^2/\lambda^2 + 1/\varphi^2} \neq 0$$

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with ε_i - indep.

$$T = \frac{n a^2}{\sigma^2 + \lambda^2} + \frac{1}{\varphi^2} \to \infty$$

$$Q \to 0$$

Calibration problem.

Idea in trivial settings.

$$y = ax + \nu$$

Take know φ

$$\psi = a\varphi + \mu$$

Heuristic ! $\hat{a} = \psi/\varphi$

and use it as a "true" a.

But: need to ~~take~~ take into account its inaccuracy.

$$y = Ax + \nu \qquad x \sim (x_0, F)$$

$$A : \mathcal{D} \to \mathcal{R} \qquad \nu \sim (0, S)$$

Have some info about A (or no info)

calibration observations:

$$\psi_i = A\varphi_i + \mu_i \qquad i = 1, \dots, K$$

K calibration measurements.

(φ_i, ψ_i) - calibration info.

$\varphi_i \in \mathcal{D}$ - known calibration signals

$\psi_i \in \mathcal{R}$ - observed result of calibration

$\mu_i \in \mathcal{R}$ - random vectors $\mu_i \sim (0, \Sigma)$ ind.

$\dim \mathcal{D} = m$

$\dim \mathcal{R} = n$

num of cal. meas $= K$.

$\Phi = [\varphi_1 \cdots \varphi_k]$ $\quad m \times k$

$\Psi = [\psi_1 \cdots \psi_k]$ $\quad n \times k$

$(\Phi, \Psi) \overset{?}{\longmapsto} (R, r)$ - optimal estimator.

calibr. info.

Separate cal. meas:

$\psi_i = A\varphi_i + \mu_i \qquad i = 1, \dots k$

$\Psi = A\Phi + M \qquad M = [\mu_1 \cdots \mu_k]$

$\times \Phi^*$ $\qquad$ random matrix

$\Psi\Phi^* = A\Phi\Phi^* + M\Phi^* \qquad EM = 0 \quad$ $n \times k$ matrix.

Assume $\Phi\Phi^*$ is invertible.

$\Phi \quad m \times \underline{K} \quad \Phi\Phi^* = m \times m$

take φ_i as indep as possible.

$\Rightarrow$ if $k \geq m$ $\quad \operatorname{rank}\Phi = m$

$\Rightarrow \Phi\Phi^*$ - invertible.

$$\Psi \underbrace{\Phi^*(\Phi\Phi^*)^{-1}}_{=\Phi^- \text{ - pseudoinverse of } \Phi} = A + M\Phi^*(\Phi\Phi^*)^{-1}$$

$$\Psi\Phi^- = A + M\Phi^-$$

$$A = \underbrace{\Psi\Phi^-}_{=A_0 \text{ - estimate of } A.} - M\Phi^-$$

$$A = A_0 + A' \qquad A' = M\Phi^-$$

$$J = E A' F A'^* = E M \underbrace{\Phi^- F \Phi^{-*}}_{=B} M^*$$

$$B = \underset{K \times m}{\Phi^*}(\Phi\Phi^*)^{-1} F (\Phi\Phi^*)^{-1} \underset{m \times k}{\Phi} \quad K \times k$$

$$J = E\, M B M^* = ?$$

$$M = [\mu_1, \dots \mu_k]$$

$$J = E[\mu_1 \dots \mu_k]\, B \begin{bmatrix} \mu_1^T \\ \vdots \\ \mu_k^T \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ - & - & - & - \\ b_{k1} & \dots & & b_{kk} \end{bmatrix}$$

$$J = E[\mu_1 \cdots \mu_k] \begin{bmatrix} \sum_i b_{1i}\mu_i^T \\ \vdots \\ \sum_i b_{ki}\mu_i^T \end{bmatrix}$$

$$= E\Big(\underbrace{\mu_1 \sum_i b_{1i}\mu_i^T}_{\sum_i b_{1i}\mu_1\mu_i^T} + \cdots + \mu_k \sum b_{ki}\mu_i^T\Big)$$

$$E\mu_j\mu_i^T = \delta_{ji}\underbrace{E\mu_i\mu_i^T}_{= S = \operatorname{var}\mu_i}$$

$$= b_{11}S + b_{22}S + \cdots + b_{kk}S$$

$$= \sum_{i=1}^{k} b_{ii} \cdot S = \operatorname{tr} B \cdot S$$

$$J = \operatorname{tr}(B)\cdot S$$

$$\operatorname{tr} B = \operatorname{tr} \Phi^*(\Phi\Phi^*)^{-1}F(\Phi\Phi^*)^{-1}\Phi$$

$$\operatorname{tr} AB^T = \sum_i (AB^T)_{ii}$$ A, B have same dim.

$$= \sum_i \sum_j A_{ij} \underbrace{B^T_{ji}}_{B_{ij}} = \sum_{ij} A_{ij} B_{ij}$$

$$= \langle A, B \rangle$$

$$\operatorname{tr} BA^T = \operatorname{tr} B^T A = \operatorname{tr} A^T B = \langle A, B \rangle$$

$$\operatorname{tr} \underset{n\times m}{A} \underset{m\times n}{C} = \operatorname{tr} CA$$

$$\operatorname{tr} B = \operatorname{tr} \Phi^* (\Phi\Phi^*)^{-1} F \underbrace{(\Phi\Phi^*)^{-1}\Phi}$$

$$= \operatorname{tr} \underbrace{(\Phi\Phi^*)^{-1}\Phi\Phi^*}_{=I} (\Phi\Phi^*)^{-1} F$$

$$= \operatorname{tr} (\Phi\Phi^*)^{-1} F = \langle (\Phi\Phi^*)^{-1}, F \rangle$$

$$A_0 = \Psi \Phi^* (\Phi\Phi^*)^{-1}$$

$$J = \langle (\Phi\Phi^*)^{-1}, F \rangle \cdot S$$

Accumulating Calibration Information.

$$\Psi \Phi^* = [\psi_1, \psi_2 .. \psi_k] \begin{bmatrix} \varphi_1^T \\ \vdots \\ \varphi_k^T \end{bmatrix}$$

$$= \sum_{i=1}^{k} \underbrace{\psi_i \varphi_i^T}_{=G_i} = \sum_{i=1}^{k} G_i = G$$

$$G_i = \psi_i \varphi_i^T$$

$$\Phi \Phi^T = \sum_{i=1}^{k} \underbrace{\varphi_i \varphi_i^T}_{= H_i} = \sum_{i=1}^{k} H_i = H$$

$$A_0 = G H^{-1}$$

$$J = \langle H^{-1}, F \rangle S$$

$$J + S = (\underbrace{\langle H^{-1}, F \rangle}_{= \alpha} + 1) S = (\alpha + 1) S$$